Solving Signomial Programs with SAGE Certificates and Partial Dualization

Riley Murray

California Institute of Technology

September 27, 2019

Joint work with Venkat Chandrasekaran and Adam Wierman (Caltech).

Nonnegativity and Optimization

Caltech

Given a function f and a set $X \subset \mathbb{R}^n$, we have

$$\begin{aligned} f_X^\star &= \inf\{f(\boldsymbol{x}) : \boldsymbol{x} \text{ in } X\} \\ &= \sup\{\gamma : f(\boldsymbol{x}) \geq \gamma \text{ for all } \boldsymbol{x} \text{ in } X\}. \end{aligned}$$

Nonnegativity and Optimization

Caltech

Given a function f and a set $X \subset \mathbb{R}^n$, we have

$$\begin{aligned} f_X^\star &= \inf\{f(\boldsymbol{x}) : \boldsymbol{x} \text{ in } X\} \\ &= \sup\{\gamma : f(\boldsymbol{x}) \geq \gamma \text{ for all } \boldsymbol{x} \text{ in } X\}. \end{aligned}$$

If we can *prove* $f - \gamma$ is ≥ 0 over X, then we know $f_X^\star \geq \gamma$.

Nonnegativity and Optimization

Caltech

Given a function f and a set $X \subset \mathbb{R}^n$, we have

$$\begin{aligned} f_X^\star &= \inf\{f(\boldsymbol{x}) : \boldsymbol{x} \text{ in } X\} \\ &= \sup\{\gamma : f(\boldsymbol{x}) \geq \gamma \text{ for all } \boldsymbol{x} \text{ in } X\}. \end{aligned}$$

If we can *prove* $f - \gamma$ is ≥ 0 over X, then we know $f_X^\star \geq \gamma$.

Obstacle: proving nonnegativity is hard (NP-Hard).

Nonnegativity and Optimization

Caltech

Given a function f and a set $X \subset \mathbb{R}^n$, we have

$$\begin{aligned} f_X^\star &= \inf\{f(\boldsymbol{x}) : \boldsymbol{x} \text{ in } X\} \\ &= \sup\{\gamma : f(\boldsymbol{x}) \geq \gamma \text{ for all } \boldsymbol{x} \text{ in } X\}. \end{aligned}$$

If we can *prove* $f - \gamma$ is ≥ 0 over X, then we know $f_X^\star \geq \gamma$.

Obstacle: proving nonnegativity is hard (NP-Hard).

Tool: tractable *sufficient conditions* for nonnegativity.

Signomials

Caltech

Signomials are functions of the form

$$\boldsymbol{x} \mapsto \sum_{i=1}^{m} c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x})$$

for real scalars c_i, and row vectors $\boldsymbol{\alpha}_i$ in $\mathbb{R}^n$.

Signomials

Caltech

Signomials are functions of the form

$$\boldsymbol{x} \mapsto \sum_{i=1}^{m} c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x})$$

for real scalars c_i, and row vectors $\boldsymbol{\alpha}_i$ in $\mathbb{R}^n$.

If $\boldsymbol{c} \geq \mathbf{0}$, then $\mathrm{Sig}(\boldsymbol{\alpha}, \boldsymbol{c})$ is convex.

Signomials

Signomials are functions of the form

$$\boldsymbol{x} \mapsto \sum_{i=1}^{m} c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x})$$

for real scalars c_i, and row vectors $\boldsymbol{\alpha}_i$ in $\mathbb{R}^n$.

If $\boldsymbol{c} \geq \mathbf{0}$, then $\mathrm{Sig}(\boldsymbol{\alpha}, \boldsymbol{c})$ is convex.

If allow arbitrary $c_i < 0$, then optimization becomes NP-Hard.

Definitions from Convex Analysis

Caltech

The **relative entropy function** is the continuous extension of

$$D(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{m} u_i \log(u_i / v_i) \quad \text{to} \quad \mathbb{R}^m_+ \times \mathbb{R}^m_+.$$

Definitions from Convex Analysis

Caltech

The **relative entropy function** is the continuous extension of

$$D(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{m} u_i \log(u_i / v_i) \quad \text{to} \quad \mathbb{R}^m_+ \times \mathbb{R}^m_+.$$

The **support function** of a convex set X is

$$\sigma_X(\boldsymbol{\lambda}) = \sup\{\boldsymbol{\lambda}^\intercal \boldsymbol{x} \,:\, \boldsymbol{x} \text{ in } X\}.$$

Definitions from Convex Analysis

Caltech

The **relative entropy function** is the continuous extension of

$$D(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{m} u_i \log(u_i / v_i) \quad \text{to} \quad \mathbb{R}^m_+ \times \mathbb{R}^m_+.$$

The **support function** of a convex set X is

$$\sigma_X(\boldsymbol{\lambda}) = \sup\{\boldsymbol{\lambda}^\intercal \boldsymbol{x} \,:\, \boldsymbol{x} \text{ in } X\}.$$

A set convex set K is called a **cone** if

$$\boldsymbol{x} \in K \quad \Rightarrow \quad \lambda \boldsymbol{x} \in K \qquad \text{for all } \lambda \geq 0;$$

Definitions from Convex Analysis

Caltech

The **relative entropy function** is the continuous extension of

$$D(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{m} u_i \log(u_i/v_i) \quad \text{to} \quad \mathbb{R}^m_+ \times \mathbb{R}^m_+.$$

The **support function** of a convex set X is

$$\sigma_X(\boldsymbol{\lambda}) = \sup\{\boldsymbol{\lambda}^\intercal \boldsymbol{x} \, : \, \boldsymbol{x} \text{ in } X\}.$$

A set convex set K is called a **cone** if

$$\boldsymbol{x} \in K \quad \Rightarrow \quad \lambda \boldsymbol{x} \in K \quad \text{for all } \lambda \geq 0;$$

the **dual cone** to K is

$$K^\dagger = \{\boldsymbol{y} \, : \, \boldsymbol{y}^\intercal \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \text{ in } K\}.$$

Theory of SAGE certificates.

The signomial X-nonnegativity cones

Caltech

Define the X-nonnegativity cone for signomials over exponents $\boldsymbol{\alpha}$:

$$C_{\mathrm{NNS}}(\boldsymbol{\alpha}, X) \doteq \{\, \boldsymbol{c} \,:\, \textstyle\sum_{i=1}^{m} c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x}) \geq 0 \text{ for all } \boldsymbol{x} \text{ in } X \}.$$

The signomial X-nonnegativity cones

Caltech

Define the X-nonnegativity cone for signomials over exponents $\boldsymbol{\alpha}$:

$$C_{\mathrm{NNS}}(\boldsymbol{\alpha}, X) \doteq \{\, \boldsymbol{c} \,:\, \textstyle\sum_{i=1}^m c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x}) \geq 0 \text{ for all } \boldsymbol{x} \text{ in } X \}.$$

These nonnegativity cones exhibit affine-invariance:

$$C_{\mathrm{NNS}}(\boldsymbol{\alpha}, X) = C_{\mathrm{NNS}}(\boldsymbol{\alpha} - \mathbf{1}\boldsymbol{u}, X) = C_{\mathrm{NNS}}(\boldsymbol{\alpha}\boldsymbol{V}, \boldsymbol{V}^{-1}X)$$

for all row vectors $\boldsymbol{u}$ in $\mathbb{R}^n$, and all invertible $\boldsymbol{V}$ in $\mathbb{R}^{n \times n}$.

X-SAGE $\Rightarrow$ X-nonnegativity

Caltech

Definition. A signomial which is nonnegative over X and which has at most one negative coefficient is an "X-AGE function."

X-SAGE $\Rightarrow$ X-nonnegativity

Caltech

Definition. A signomial which is nonnegative over X and which has at most one negative coefficient is an "X-AGE function."

We take sums of X-AGE cones to obtain the X-**SAGE cone**

$$C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X) = \sum_{k=1}^{m} \underbrace{\{\boldsymbol{c} \;:\; \boldsymbol{c}_{\setminus k} \geq \mathbf{0} \text{ and } \boldsymbol{c} \text{ in } C_{\mathrm{NNS}}(\boldsymbol{\alpha}, X)\}}_{k^{\text{th}}\ X\text{-AGE cone}}.$$

X-SAGE $\Rightarrow$ X-nonnegativity

Caltech

Definition. A signomial which is nonnegative over X and which has at most one negative coefficient is an "X-AGE function."

We take sums of X-AGE cones to obtain the X-**SAGE cone**

$$C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X) = \sum_{k=1}^{m} \underbrace{\{\boldsymbol{c} \;:\; \boldsymbol{c}_{\setminus k} \geq \mathbf{0} \text{ and } \boldsymbol{c} \text{ in } C_{\mathrm{NNS}}(\boldsymbol{\alpha}, X)\}}_{k^{\text{th}}\ X\text{-AGE cone}}.$$

Crucial question: How to represent the AGE cones?

The convex duality behind AGE cones

Caltech

Fix $\boldsymbol{\alpha}$ in $\mathbb{R}^{m \times n}$, and $\boldsymbol{c}$ in $\mathbb{R}^m$ satisfying $\boldsymbol{c}_{\setminus k} \geq \mathbf{0}$. Convex $X \subset \mathbb{R}^n$.

The convex duality behind AGE cones

Caltech

Fix $\boldsymbol{\alpha}$ in $\mathbb{R}^{m \times n}$, and $\boldsymbol{c}$ in $\mathbb{R}^m$ satisfying $\boldsymbol{c}_{\setminus k} \geq \mathbf{0}$. Convex $X \subset \mathbb{R}^n$.

Appeal to affine invariance of $C_{\text{NNS}}(\boldsymbol{\alpha}, X)$, and rearrange terms:

$$\sum_{i=1}^m c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x}) \geq 0 \;\Leftrightarrow\; \sum_{i=1}^m c_i \exp([\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_k] \cdot \boldsymbol{x}) \geq 0$$

$$\sum_{i \neq k} c_i \exp([\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_k] \cdot \boldsymbol{x}) \geq -c_k.$$

The convex duality behind AGE cones

Caltech

Fix $\boldsymbol{\alpha}$ in $\mathbb{R}^{m \times n}$, and $\boldsymbol{c}$ in $\mathbb{R}^m$ satisfying $\boldsymbol{c}_{\setminus k} \geq \mathbf{0}$. Convex $X \subset \mathbb{R}^n$.

Appeal to affine invariance of $C_{\mathrm{NNS}}(\boldsymbol{\alpha}, X)$, and rearrange terms:

$$\sum_{i=1}^m c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x}) \geq 0 \;\Leftrightarrow\; \sum_{i=1}^m c_i \exp([\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_k] \cdot \boldsymbol{x}) \geq 0$$

$$\sum_{i \neq k} c_i \exp([\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_k] \cdot \boldsymbol{x}) \geq -c_k.$$

Appeal to convex duality. The nonnegativity condition

$$\inf \left\{ \sum_{i \neq k} c_i \exp([\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_k] \cdot \boldsymbol{x}) \;:\; \boldsymbol{x} \text{ in } X \right\} \geq -c_k$$

holds **if and only if**

The convex duality behind AGE cones

Caltech

Fix $\boldsymbol{\alpha}$ in $\mathbb{R}^{m \times n}$, and $\boldsymbol{c}$ in $\mathbb{R}^m$ satisfying $\boldsymbol{c}_{\setminus k} \geq \mathbf{0}$. Convex $X \subset \mathbb{R}^n$.

Appeal to affine invariance of $C_{\mathrm{NNS}}(\boldsymbol{\alpha}, X)$, and rearrange terms:

$$\begin{aligned} \textstyle\sum_{i=1}^m c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x}) \geq 0 \;\Leftrightarrow\; & \textstyle\sum_{i=1}^m c_i \exp([\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_k] \cdot \boldsymbol{x}) \geq 0 \\ & \textstyle\sum_{i \neq k} c_i \exp([\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_k] \cdot \boldsymbol{x}) \geq -c_k. \end{aligned}$$

Appeal to convex duality. The nonnegativity condition

$$\inf \left\{ \textstyle\sum_{i \neq k} c_i \exp([\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_k] \cdot \boldsymbol{x}) \;:\; \boldsymbol{x} \text{ in } X \right\} \geq -c_k$$

holds **if and only if** there exists $\boldsymbol{\nu}$ in $\mathbb{R}^{m-1}$, $\boldsymbol{\lambda}$ in $\mathbb{R}^n$ satisfying

The convex duality behind AGE cones

Caltech

Fix $\boldsymbol{\alpha}$ in $\mathbb{R}^{m \times n}$, and $\boldsymbol{c}$ in $\mathbb{R}^m$ satisfying $\boldsymbol{c}_{\setminus k} \geq \mathbf{0}$. Convex $X \subset \mathbb{R}^n$.

Appeal to affine invariance of $C_{\mathrm{NNS}}(\boldsymbol{\alpha}, X)$, and rearrange terms:

$$\textstyle\sum_{i=1}^m c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x}) \geq 0 \;\Leftrightarrow\; \sum_{i=1}^m c_i \exp([\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_k] \cdot \boldsymbol{x}) \geq 0$$

$$\textstyle\sum_{i \neq k} c_i \exp([\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_k] \cdot \boldsymbol{x}) \geq -c_k.$$

Appeal to convex duality. The nonnegativity condition

$$\inf \left\{ \textstyle\sum_{i \neq k} c_i \exp([\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_k] \cdot \boldsymbol{x}) \;:\; \boldsymbol{x} \text{ in } X \right\} \geq -c_k$$

holds **if and only if** there exists $\boldsymbol{\nu}$ in $\mathbb{R}^{m-1}$, $\boldsymbol{\lambda}$ in $\mathbb{R}^n$ satisfying

$$\sigma_X(\boldsymbol{\lambda}) + D(\boldsymbol{\nu}, \boldsymbol{c}_{\setminus k}) - \boldsymbol{\nu}^\intercal \mathbf{1} \leq c_k, \text{ and}$$

$$[\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k]\boldsymbol{\nu} + \boldsymbol{\lambda} = \mathbf{0}.$$

X is tractable $\Rightarrow$ X-SAGE is tractable

Caltech

There are two constraints in an AGE cone:

- $[\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k]\boldsymbol{\nu} + \boldsymbol{\lambda} = \mathbf{0}$ — definitely tractable
- $\sigma_X(\boldsymbol{\lambda}) + D(\boldsymbol{\nu}, \boldsymbol{c}_{\setminus k}) - \boldsymbol{\nu}^\intercal \mathbf{1} \leq c_k$ — tractability unclear

X is tractable $\Rightarrow$ X-SAGE is tractable

Caltech

There are two constraints in an AGE cone:

- $[\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k]\boldsymbol{\nu} + \boldsymbol{\lambda} = \mathbf{0}$ definitely tractable
- $\sigma_X(\boldsymbol{\lambda}) + D(\boldsymbol{\nu}, \boldsymbol{c}_{\setminus k}) - \boldsymbol{\nu}^\intercal \mathbf{1} \leq c_k$ tractability unclear

Support function $\sigma_X(\boldsymbol{\lambda}) = \sup\{\boldsymbol{\lambda}^\intercal \boldsymbol{x} : \boldsymbol{x} \text{ in } X\}$ always convex.

When is it tractable?

X is tractable $\Rightarrow$ X-SAGE is tractable

Caltech

There are two constraints in an AGE cone:

- $[\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k]\boldsymbol{\nu} + \boldsymbol{\lambda} = \mathbf{0}$ definitely tractable
- $\sigma_X(\boldsymbol{\lambda}) + D(\boldsymbol{\nu}, \boldsymbol{c}_{\setminus k}) - \boldsymbol{\nu}^\intercal \mathbf{1} \leq c_k$ tractability unclear

Support function $\sigma_X(\boldsymbol{\lambda}) = \sup\{\boldsymbol{\lambda}^\intercal \boldsymbol{x} : \boldsymbol{x} \text{ in } X\}$ always convex.

When is it tractable?

$$X = \mathbb{R}^n \quad \Rightarrow \quad \sigma_X(\boldsymbol{\lambda}) = \begin{cases} 0 & \text{if } \boldsymbol{\lambda} = \mathbf{0} \\ +\infty & \text{if } \boldsymbol{\lambda} \neq \mathbf{0} \end{cases}$$

X is tractable $\Rightarrow$ X-SAGE is tractable

Caltech

There are two constraints in an AGE cone:

- $[\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k]\boldsymbol{\nu} + \boldsymbol{\lambda} = \mathbf{0}$ — definitely tractable
- $\sigma_X(\boldsymbol{\lambda}) + D(\boldsymbol{\nu}, \boldsymbol{c}_{\setminus k}) - \boldsymbol{\nu}^\intercal \mathbf{1} \leq c_k$ — tractability unclear

Support function $\sigma_X(\boldsymbol{\lambda}) = \sup\{\boldsymbol{\lambda}^\intercal \boldsymbol{x} : \boldsymbol{x} \text{ in } X\}$ always convex.

When is it tractable?

$$X = \{\boldsymbol{x} : \|\boldsymbol{x} - \boldsymbol{a}\| \leq r\} \quad \Rightarrow \quad \sigma_X(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^\intercal \boldsymbol{a} + r\|\boldsymbol{\lambda}\|_*$$

X is tractable $\Rightarrow$ X-SAGE is tractable

Caltech

There are two constraints in an AGE cone:

- $[\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k]\boldsymbol{\nu} + \boldsymbol{\lambda} = \mathbf{0}$ — definitely tractable
- $\sigma_X(\boldsymbol{\lambda}) + D(\boldsymbol{\nu}, \boldsymbol{c}_{\setminus k}) - \boldsymbol{\nu}^\intercal \mathbf{1} \leq c_k$ — tractability unclear

Suppose $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \in K\}$ is strictly feasible.

Then

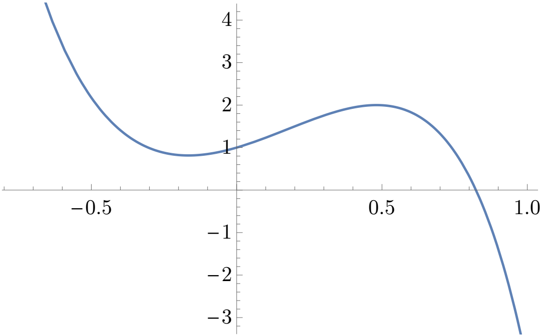
$$\sigma_X(\boldsymbol{\lambda}) \leq t$$

if and only if some $\boldsymbol{\eta}$ satisfies

$$\boldsymbol{\eta} \in K^\dagger, \qquad \boldsymbol{A}^\intercal \boldsymbol{\eta} + \boldsymbol{\lambda} = \mathbf{0}, \quad \text{and} \quad \boldsymbol{b}^\intercal \boldsymbol{\eta} \leq t.$$

A univariate example

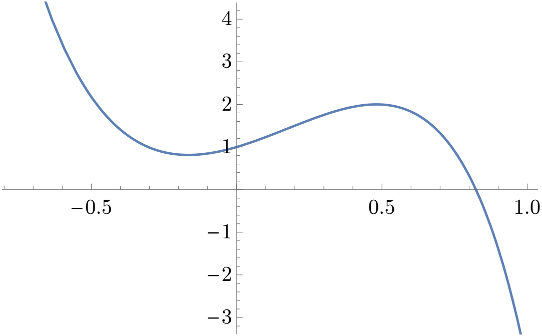
$$f(x) = e^{-3x} + e^{-2x} + 4e^{x} + e^{2x} - 4e^{-x} - 1 - e^{3x} \text{ over } x \leq 0$$



A univariate example

Caltech

$$f(x) = e^{-3x} + e^{-2x} + 4e^{x} + e^{2x} - 4e^{-x} - 1 - e^{3x} \text{ over } x \leq 0$$



$$f_1(x) = 0.88 \cdot e^{-3x} + 0.82 \cdot e^{-2x} + 2.69 \cdot e^{x} + 0.12 \cdot e^{2x} - 4 \cdot e^{-x}$$

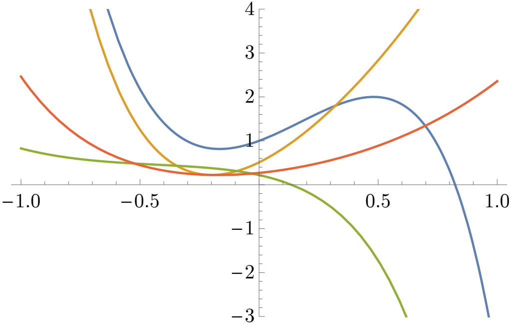
$$f_2(x) = 0.10 \cdot e^{-3x} + 0.15 \cdot e^{-2x} + 0.90 \cdot e^{x} + 0.12 \cdot e^{2x} - 1$$

$$f_3(x) = 0.02 \cdot e^{-3x} + 0.03 \cdot e^{-2x} + 0.41 \cdot e^{x} + 0.76 \cdot e^{2x} - e^{3x}$$

A univariate example

Caltech

$$f(x) = e^{-3x} + e^{-2x} + 4e^{x} + e^{2x} - 4e^{-x} - 1 - e^{3x} \text{ over } x \leq 0$$



$$f_1(x) = 0.88 \cdot e^{-3x} + 0.82 \cdot e^{-2x} + 2.69 \cdot e^{x} + 0.12 \cdot e^{2x} - 4 \cdot e^{-x}$$
$$f_2(x) = 0.10 \cdot e^{-3x} + 0.15 \cdot e^{-2x} + 0.90 \cdot e^{x} + 0.12 \cdot e^{2x} - 1$$
$$f_3(x) = 0.02 \cdot e^{-3x} + 0.03 \cdot e^{-2x} + 0.41 \cdot e^{x} + 0.76 \cdot e^{2x} - e^{3x}$$

Geometric-form signomials

Caltech

If $\boldsymbol{x} > 0$, then

$$\boldsymbol{x} \mapsto \sum_{i=1}^{m} c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i}$$

is defined for any real $\boldsymbol{\alpha}_i$.

If $X \subset \mathbb{R}^n_{++}$, then

$$\{\boldsymbol{c} : \sum_{i=1}^{m} c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i} \geq 0 \text{ for all } \boldsymbol{x} \text{ in } X\} = C_{\mathrm{NNS}}(\boldsymbol{\alpha}, \log X).$$

Geometric-form signomials

Caltech

If $\boldsymbol{x} > 0$, then

$$\boldsymbol{x} \mapsto \sum_{i=1}^{m} c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i}$$

is defined for any real $\boldsymbol{\alpha}_i$.

If $X \subset \mathbb{R}^n_{++}$, then

$$\{\boldsymbol{c} : \sum_{i=1}^{m} c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i} \geq 0 \text{ for all } \boldsymbol{x} \text{ in } X\} = C_{\mathrm{NNS}}(\boldsymbol{\alpha}, \log X).$$

Therefore

$$C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, \log X) \subset \{\boldsymbol{c} : \sum_{i=1}^{m} c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i} \geq 0 \text{ for all } \boldsymbol{x} \text{ in } X\},$$

and the L.H.S. inherits tractability from $Y = \log X$.

Optimization.

Simple SAGE relaxations

Caltech

Consider $f(\boldsymbol{x}) = \sum_{i=1}^{m} c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x})$ with $\boldsymbol{\alpha}_1 = \mathbf{0}$. Fix convex X.

The primal and dual SAGE relaxations for $f_X^\star$ are

$$
\begin{aligned}
f_X^{\mathrm{SAGE}} &= \sup\{\, \gamma \,:\, \boldsymbol{c} - \gamma(1, 0, \ldots, 0) \text{ in } C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)\} \\
&= \inf\{\boldsymbol{c}^\intercal \boldsymbol{v} \,:\, v_1 = 1 \text{ and } \boldsymbol{v} \text{ in } C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)^\dagger\}.
\end{aligned}
$$

Simple SAGE relaxations

Caltech

Consider $f(\boldsymbol{x}) = \sum_{i=1}^{m} c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x})$ with $\boldsymbol{\alpha}_1 = \boldsymbol{0}$. Fix convex X.

The primal and dual SAGE relaxations for $f_X^\star$ are

$$\begin{aligned} f_X^{\mathrm{SAGE}} &= \sup\{\, \gamma \,:\, \boldsymbol{c} - \gamma(1, 0, \ldots, 0) \text{ in } C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)\} \\ &= \inf\{\boldsymbol{c}^\intercal \boldsymbol{v} \,:\, v_1 = 1 \text{ and } \boldsymbol{v} \text{ in } C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)^\dagger\}. \end{aligned}$$

The dual X-SAGE cone can be expressed as

$$\begin{aligned} C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)^\dagger = \mathrm{cl}\{\boldsymbol{v} : \; & \text{some } \boldsymbol{z}_1, \ldots, \boldsymbol{z}_m \text{ in } \mathbb{R}^n \text{ satisfy} \\ & \ldots \text{ relative entropy constraints } \ldots \\ & \text{and } \boldsymbol{z}_k / v_k \in X \text{ for all } k \text{ in } [m]\}. \end{aligned}$$

Simple SAGE relaxations

Caltech

Consider $f(\boldsymbol{x}) = \sum_{i=1}^{m} c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x})$ with $\boldsymbol{\alpha}_1 = \mathbf{0}$. Fix convex X.

The primal and dual SAGE relaxations for $f_X^\star$ are

$$\begin{aligned} f_X^{\mathrm{SAGE}} &= \sup\{\, \gamma \,:\, \boldsymbol{c} - \gamma(1,0,\ldots,0) \text{ in } C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)\} \\ &= \inf\{\boldsymbol{c}^\intercal \boldsymbol{v} \,:\, v_1 = 1 \text{ and } \boldsymbol{v} \text{ in } C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)^\dagger\}. \end{aligned}$$

The dual X-SAGE cone can be expressed as

$$\begin{aligned} C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)^\dagger = \mathrm{cl}\{\boldsymbol{v} : &\text{ some } \boldsymbol{z}_1, \ldots, \boldsymbol{z}_m \text{ in } \mathbb{R}^n \text{ satisfy} \\ &\ldots \text{ relative entropy constraints } \ldots \\ &\text{and } \boldsymbol{z}_k / v_k \in X \text{ for all } k \text{ in } [m]\}. \end{aligned}$$

Solution recovery? Consider vectors $\boldsymbol{x}_k = \boldsymbol{z}_k / v_k$ for k in $[m]$.

A small example

Caltech

$$\inf_{\boldsymbol{x}\in\mathbb{R}^3} \; f(\boldsymbol{x}) \doteq 0.5\exp(x_1 - x_2) - \exp x_1 - 5\exp(-x_2)$$

$$\text{s.t.} \;\; \exp(x_2 - x_3) + \exp x_2 + 0.05\exp(x_1 + x_3) \leq 100$$

$$\log 70 \leq x_1 \leq \log 150$$

$$\log 1.0 \leq x_2 \leq \log 30$$

$$\log 0.5 \leq x_3 \leq \log 21$$

A small example

Caltech

$$
\begin{aligned}
\inf_{\boldsymbol{x}\in\mathbb{R}^3} \; & f(\boldsymbol{x}) \doteq 0.5\exp(x_1 - x_2) - \exp x_1 - 5\exp(-x_2) \\
\text{s.t.} \; & \exp(x_2 - x_3) + \exp x_2 + 0.05\exp(x_1 + x_3) \leq 100 \\
& \log 70 \leq x_1 \leq \log 150 \\
& \log 1.0 \leq x_2 \leq \log 30 \\
& \log 0.5 \leq x_3 \leq \log 21
\end{aligned}
$$

Compute $f_X^{\mathrm{SAGE}} = -147.85713 \leq f_X^\star$, recover feasible

$$\boldsymbol{x}^\star = (5.01063529,\; 3.40119660, -0.48450710)$$

satisfying $f(\boldsymbol{x}^\star) = -147.66666$. *This is actually optimal!*

Nonconvex contraints

Caltech

Q: What should we do when some constraints are nonconvex?

A: Combine X-SAGE certificates with Lagrangian relaxations.

Concretely, suppose we want to minimize f over

$$\Omega \doteq X \cap \{\boldsymbol{x} \,:\, g(\boldsymbol{x}) \leq \mathbf{0}\}$$

where X is convex, but $g_1, \ldots, g_k$ are nonconvex signomials.

Then, if $\lambda_1, \ldots, \lambda_k$ are nonnegative dual variables, we have

$$\inf_{\boldsymbol{x} \in \Omega} f(\boldsymbol{x}) \geq \sup \left\{ \gamma \,:\, f + \sum_{i=1}^{k} \lambda_i g_i - \gamma \text{ is } X\text{-SAGE} \right\}.$$

A bigger example

Caltech

$$
\begin{aligned}
\inf\ & 720H_c + 43200\varphi + 14400\varphi^3 + 5760\varphi^5 + R^2\varphi^3 + 0.4R^2\varphi^5 - 7198.2 \qquad \text{(Ex11)} \\
\text{s.t. } & 15 \leq H \leq 25, \quad 15 \leq H_c \leq 25, \quad 12 \leq H_t \leq 19 \\
& 330 \leq R \leq 380, \quad 330 \leq R_M \leq 380, \quad 0.05 \leq \varphi \leq 0.2 \\
& 252.154H^{-2} + 4500R^{-2} \leq 1, \ R^{-1}R_M - 0.5HR^{-1} = 1 \\
& 0.0125H + 0.00833R\varphi + 0.0000694R\varphi^5 - 0.001389R\varphi^3 \leq 1 \\
& 30.52132H_c^{-1} - 120H_c^{-1}\varphi - 40H_c^{-1}\varphi^3 - 16H_c^{-1}\phi^5 \leq 1 \\
& 2238.432H_c^{-3} + 53720.208H_c^{-4}\varphi + 17906.736H_c^{-4}\varphi^3 + 7162.694H_c^{-4}\varphi^5 \\
& \quad + 19.995H_c^{-1} - 8951.297H_c^{-4} - 120H_c^{-1}\varphi - 40H_c^{-1}\varphi^3 - 16H_c^{-1}\varphi^5 \leq 1 \\
& 252.1543H_t^{-2} + 0.005837H_t^{-2}R^2\varphi^4 + 4500R^{-2} - 0.0175H_t^{-2}R^2\varphi^2 \\
& \quad - 0.000778H_t^{-2}R^2\varphi^6 \leq 1 \\
& 67.73085H^{-1.8}R_M^{0.2}\varphi^{0.2} + 146.53487H^{-0.8}R_M^{-0.8}\varphi^{0.2} \\
& \quad + 393.09732H^{0.2}R_M^{-1.8}\varphi^{0.2} \leq 1 \\
& HH_t^{-1} + 0.5H_t^{-1}R\phi^2 + 0.02777H_t^{-1}R\phi^3 - 0.0416667H_t^{-1}R\phi^4 \\
& \quad - 0.16663H_t^{-1}R\varphi - 0.001389H_t^{-1}R\varphi^5 = 1 \\
& 2HR^{-1}\varphi^{-2} - 2H_cR^{-1}\varphi^{-2} - 0.41667\varphi^2 - 0.16944\varphi^4 = 1
\end{aligned}
$$

Benchmark problem from 1970's. SAGE set a new record.

Use the `sageopt` python package.

Caltech

- Python 3.5 or higher (recommend $\geq$ 3.6).
- "`pip install sageopt`"
- Signomial and polynomial optimization.
- Require open-source convex solver, ECOS.
- Recommend commercial solver, MOSEK.

Concluding Remarks

The content of this presentation is a small fraction of

Signomial and Polynomial Optimization via Relative Entropy and Partial Dualization

– a paper by R.M., Venkat Chandraksekaran, and Adam Wierman.